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Generalization of the Darwin Lagrangian

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Starting from a Lagrangian treatment of classical electrodynamics and using simple heuristic arguments, an effective generalization of the Darwin Lagrangian for point charges moving with arbitrary velocities ($v < c$) is obtained. In addition, another, the simplest *a priori* Lagrangian (preserving the most important features of standard classical theory) is proposed.

Standard classical electrodynamics rest on the two "pillars": Maxwell's equations (for fields and their sources) and Lorentz-Dirac equations of motion (for charged particles in given electromagnetic fields). Formally, one can interpret the contents of these fundamental groups of equations in three different ways: (1) one can regard the Lorentz-Dirac equations as formal definitions of fields **E** and **B**, and Maxwell equations as a genuine physical law; or (2) one can regard the Maxwell equations as implicit definitions of fields via positions and motions of charges, and the Lorentz-Dirac equations as a physical law with an empirical content; or (3) one can regard both fundamental groups of equations on an equal footing as semianalytical and semiempirical laws which can be modified as a result of experimental or theoretical arguments. Keeping in mind the history of physics, we think that the last approach is the best one. At present, however, none of these approaches gives a straightforward formal connection between these pillars. Hence, from a methodological point of view, standard classical electrodynamics is an essentially dualistic theory: not strictly a field theory and not a theory of a direct electromagnetic interaction of charged particles.

A second deficiency of standard classical electrodynamics is connected with an essential ineffectiveness of this theory, even in the treatment of the very fundamental problem of the motion of charges. Strictly speaking, one cannot write down the concrete equation of motion for two charges (of

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comparable masses) if one does not know the previous full space-time trajectories of these particles.

There have been definite attempts to overcome this particle–field dualism, and also to overcome the essential ineffectiveness of the standard classical theory. One such attempt is the elegant theoretical program [developed by Fokker (1929), Wheeler and Feynman (1945), and Hoyle and Narlikar (1974)] in which the electromagnetic field plays the intermediary for the interaction between charged particles, and can be eliminated completely. While this is recognised as a very interesting approach, unfortunately, the particular theory elaborated so far exhibits the same kind of ineffectiveness as the standard version of classical electrodynamics. Other proposals aiming to overcome the mentioned ineffectiveness start as a rule with the so-called *a priori* Lagrangians deprived of any heuristic connection with the Maxwell equations.

A quite different attempt is traditionally connected with a program of nonlinear field theories, which (in view of serious mathematical complexity) lead also to very ineffective proposals.

The aim of this paper is to give a concrete and simple possibility of "throwing a bridge" across the gap between Maxwell's equations and equations of motion, in order to obtain in the final result an effective dynamical theory of electromagnetic interaction of charged point particles, characterized by a lagrangian dependent on the instantaneous positions and velocities of these particles.

We start from the Lagrangian treatment of the Maxwell equations (within field-theoretic treatment) in order to obtain a proper Lagrangian for charged point particles moving with arbitrary velocities ($v < c$).

The basic integral invariant I of Noether's theorem is

$$I = \frac{1}{c}\int \mathscr{L}\, d^4x = \int \mathscr{L}\, d^3x \cdot dt = \int L\, dt \tag{1}$$

where the functional $\mathscr{L}$ must be the Lagrangian density, which, via Hamilton's principle of least action, yields the Maxwell electromagnetic field equations. It is well known (e.g., Rohrlich, 1965; Jackson, 1975) that the Maxwell equations can be obtained equally well from the two possible options of the Lagrangian density

$$\mathscr{L}_1 = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} j_\mu A^\mu \tag{2}$$

or

$$\mathscr{L}_2 = -\frac{1}{8\pi} \partial_\mu A_\gamma\, \partial^\mu A^\nu - \frac{1}{c} j_\mu A^\mu \tag{3}$$

which may be supplemented by the Lorentz condition

$$\partial_\mu A^\mu = 0 \tag{4}$$

and the connection between the field strengths and the potentials:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \qquad F^{\nu\mu} = -F^{\mu\nu} \tag{5}$$

where

$$F^{\mu\nu} = \begin{pmatrix} 0, & -E_x, & -E_y, & -E_z \\ E_x, & 0, & -B_z, & B_y \\ E_y, & B_z, & 0, & -B_x \\ Ez, & -B_y, & B_x, & 0 \end{pmatrix},$$

$$F_{\mu\nu} = g_{\mu\alpha} F^{\alpha\beta} g_{\beta\nu} = \begin{pmatrix} 0, & E_x, & E_y, & E_z \\ -E_x, & 0, & -B_z, & B_y \\ -E_y, & B_z, & 0, & -B_x \\ -E_z, & -B_y, & B_x, & 0 \end{pmatrix} \tag{5'}$$

and where $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$, according to the more frequently used convention for the metric tensor, as used by Jackson (1975), but not by Rohrlich (1965).

The principle of least action based on the action integral with $\mathscr{L}$ given by (2) yields the Euler–Lagrange equations, which become the Maxwell field equations:

$$\partial_\mu F^{\mu\nu} = +\frac{4\pi}{c} j^\nu \tag{6}$$

and here we have the advantage that Maxwell equations in this form do not require the additional specification of the Lorentz condition (4).

An alternative choice of $\mathscr{L}$ given by (3) also yields the Maxwell field equations in the form

$$\Box A^\mu = +\frac{4\pi}{c} j^\mu \tag{7}$$

which, supplemented by (4) and (5), are equivalent to the Maxwell equations (6).

Now, what is essential to the method presented here is that we insist that there is a concrete specification of $\mathscr{L}$, some linear combination of (2) and (3), which not only yields (in the usual way) the Maxwell equations when $\mathscr{L}$ is treated as a function of A^ν and $\partial^\mu A^\nu$, but also represents the "true" Lagrangian density in a sense that it also yields the equation of motion for field sources, i.e., for charges, when

$$L = \int \mathscr{L} \, d^3x \tag{8}$$

is treated as a function of the positions and velocities of charges. Hence, a proper specification of $\mathscr{L}$ and L can give not only necessary relations between sources and their fields (which are expressed by Maxwell equations), but, above all, also can give the desired form of effective equations for charged point particles.

Now we must solve the following crucial problem: what is the exact form of fields (connected with their particle sources) that must be substituted into $\mathscr{L}$, given primarily in terms of A^ν and $\partial^\mu A^\nu$? And here (starting from the observation that the most experimentally confirmed domain of electrodynamics is connected with stationary and uniform motions of charges) we can postulate (according to the essential spirit of the classical Lagrangian formalism) that (8) must contain instantaneous positions and velocities of charged particles only (but not their accelerations or higher derivatives of their positions). This natural postulate gives us immediately a solution of the considered problem, because the only fields that conform to this condition are the well-known "velocity fields" of the considered charges, i.e., fields corresponding in standard theory to uniform motions of our charges. For actual (nonuniform) motions of the considered charges such fields cannot be the proper fields appearing in Lorentz–Dirac equations of motion, but they can be the proper "ghost fields" introduced for the evaluation of $\mathscr{L}$ only. Now, fields are not connected simply with forces, but are connected with energetic characteristics of the electromagnetic system.

According to our postulate, we have

$$A_i^\nu = \frac{Q_i}{R_i[1-\beta_i^2+(\boldsymbol{\beta}_i \hat{\mathbf{R}}_i)^2]^{1/2}} [1, \boldsymbol{\beta}_i]$$

$$\mathbf{E}_i = \frac{Q_i(1-\beta_i^2)\hat{\mathbf{R}}_i}{R_i^2[1-\beta_i^2+(\boldsymbol{\beta}_i \hat{\mathbf{R}}_i)^2]^{3/2}}, \qquad \mathbf{B}_i = \boldsymbol{\beta}_i \times \mathbf{E}_i$$

where $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$, $R_i = |\mathbf{R}_i|$, $\hat{\mathbf{R}}_i = \mathbf{R}_i / R_i$, $\boldsymbol{\beta}_i = \mathbf{v}_i / c$, and $\mathbf{r}_i$ and $\mathbf{v}_i$ are the actual position and velocity of Q_i.

Substituting these velocity fields into (8), we obtain for two charges Q_i, Q_j

$$L_{1ij} = \int \mathscr{L}_{1ij} \, d^3x \tag{9}$$

$$L_{2ij} = \int \mathcal{L}_{2ij}\, d^3x \tag{10}$$

where L_{ij} are those parts of the full Lagrangian L that correspond to the interaction between Q_i and Q_j. Following the well-known "philosophy" of Wheeler and Feynman, only these L_{ij} terms can be treated seriously for point charges, while the self-terms L_{ii} (divergent for point charges) are treated in this method as semiempirical terms of the form $-m_{0i}c^2(1-v_i^2/c^2)^{1/2}$.

Integrals (9) and (10), unfortunately, do not seem to be analytically computable, but we can make a proper choice, i.e., to find a proper linear combination of (9) and (10), if v^2/c^2 approximations of these integrals (which can be easily calculated) are compared with the well-known Darwin approximation of the same type.

Corresponding calculations give us the following results:

$$L_{1ij} = -\frac{Q_iQ_j}{r}[1+\tfrac{1}{2}\beta^2-\tfrac{1}{2}(\boldsymbol{\beta}\hat{\mathbf{r}})^2-\tfrac{1}{2}\boldsymbol{\beta}_i\boldsymbol{\beta}_j-\tfrac{1}{2}(\boldsymbol{\beta}_i\hat{\mathbf{r}})(\boldsymbol{\beta}_i\hat{\mathbf{r}})] \tag{11}$$

$$L_{2ij} = -\frac{Q_iQ_j}{r}[1+\beta^2-(\boldsymbol{\beta}\hat{\mathbf{r}})^2-\tfrac{1}{2}\boldsymbol{\beta}_i\boldsymbol{\beta}_j-\tfrac{1}{2}(\boldsymbol{\beta}_i\hat{\mathbf{r}})(\boldsymbol{\beta}_j\hat{\mathbf{r}})] \tag{12}$$

while the Darwin Lagrangian L_D is

$$L_{Dij} = -\frac{Q_iQ_j}{r}[1-\tfrac{1}{2}\boldsymbol{\beta}_i\boldsymbol{\beta}_j-\tfrac{1}{2}(\boldsymbol{\beta}_i\hat{\mathbf{r}})(\boldsymbol{\beta}_j\hat{\mathbf{r}})] \tag{13}$$

where $r=|\mathbf{r}|$, $\hat{\mathbf{r}}=\mathbf{r}/r$, $\mathbf{r}=\mathbf{r}_j-\mathbf{r}_i$, $\boldsymbol{\beta}=\boldsymbol{\beta}_j-\boldsymbol{\beta}_i$.

Now it is evident that if we want to obtain the simplest Lagrangian that is a generalization of (13), according to the adopted heuristic method, we must take the following form of L:

$$L_{3ij} = \int \mathcal{L}_{3ij}\, d^3x \tag{14}$$

where

$$\mathcal{L}_{3ij} = 2\mathcal{L}_{1ij} = \mathcal{L}_{2ij} \tag{15}$$

Even in the case of the nonexistence of an analytical solution of this integral, it is evident that by assuming the Lagrangian (14), we obtain an effective theory of the electromagnetic interaction which can be tested experimentally.

Apart from the obtained result (based on heuristic considerations given above), we also want to present here another proposal for the hypothetical

Lagrangian, which is the simplest relativistic generalization of the Darwin Lagrangian:

$$L_{4ij} = -\frac{Q_i Q_j (1 - \boldsymbol{\beta}_i \boldsymbol{\beta}_j)}{r_{ij}[1 - \boldsymbol{\beta}_i \boldsymbol{\beta}_j + (\boldsymbol{\beta}_i \hat{\mathbf{r}})(\boldsymbol{\beta}_j \hat{\mathbf{r}})]^{1/2}} \tag{16}$$

This Lagrangian can be treated as an *a priori* one, which preserves all essential features of the standard classical electrodynamics interpreted in the spirit of the Wheeler–Feynman theoretical program, but at the same time it is more effective than (14) because of its simple analytical form.

Assuming the Lagrangian (14) or (16), we proclaim a formal transition of pragmatic methods of classical electrodynamics from the domain of partial differential equations to the essentially simpler domain of ordinary differential equations.

Naturally, one can ask for the reason for making the above hypotheses. The main reason is that an extensive and detailed analysis of the experimental basis of electrodynamics shows that we still have too narrow a range of experimental data to insist that the complicated and essentially ineffective standard theory is the unique proper theory of electromagnetic interaction in its classical limit. The nonstandard theories (such as the Wheeler–Feynman theoretical program) and well-known difficulties of the standard theory in the area of radiation phenomena show that even in the well-defined domain of classical problems the standard theory can be replaced by an essentially different and better theory. Furthermore, we know that today quantum electrodynamics cannot uniquely determine (starting from fundamental principles only) all the important features of the electromagnetic interaction of two moving charges in the classical limit. Hence, any theoretical argument that exploits a connection between the Maxwell equations and the equations of motion of charged particles is of interest. A serious modification of classical electrodynamics is worth considering in view of the new light cast by experimental evidence (e.g., Kunstatter and Trainor, 1984) on the historical rivalry of local and nonlocal theories of basic physical interactions.

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